Root Locus

Visualizing Characteristic Roots in Term of the Controller Gain

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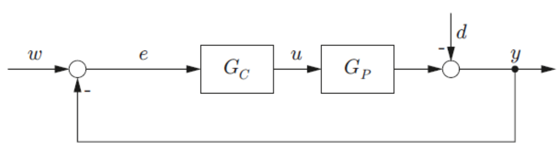
PURPOSE OF THE METHOD

The approach visualizes the characteristic roots of a closed loop system in terms of the controller gain. Therefore, it offers an intuitive controller tuning method, in order to match individual performance criteria.

PROBLEM STATEMENT

The root locus method was developed by W. Evans in 1948 [1,2]. It is a calculation approach for computing and visualizing the closed-loop eigenvalues as a function of the open-loop gain. It offers a possibility to study the stability as well as the performance of a system. Today, it is a classical tool for system synthesis and analysis of LTI systems. In the following the standard SISO control loop is used.



In the following, a linear time-invariant system $G_P(s) = A(s)/B(s)$ and a controller $G_C(s,\boldsymbol{k})$ with the controller parameter $\boldsymbol{k}$ is considered. Therefore, the following characteristic equation results

$$\delta(s,\boldsymbol{k}) = 1 + G_0 = 1 + G_P(s)G_C(s,\boldsymbol{k}) = 0$$

The location of the zeros of $\delta(s,\boldsymbol{k})$ is a function of the open-loop controller gain $k(\boldsymbol{k})$. The family of all zeros for a given interval of the open-loop gain $k(\boldsymbol{k})$ is named the root locus. For $k(\boldsymbol{k}) > 0$ it is the primary and for $k(\boldsymbol{k}) < 0$ the complementary root locus. Starting point for the root locus calculation is the modified characteristic equation

$$G_P(s)G_C(s,\boldsymbol{k}) = G_0(s) = k(\boldsymbol{k})\hat{G}_0\ (s) = -1$$

After substituting $s = \sigma + j\omega$ the equation can be split into a magnitude and a phase equation.

The **phase condition** is used to find the points in the complex domain which belong to the root locus:

$$\varphi\ (k(\boldsymbol{k})) = \varphi(-1/\hat{G}_0(s))$$

with $\varphi(k(\boldsymbol{k})) = 0°$ and $\varphi(1) = 0°$ follows

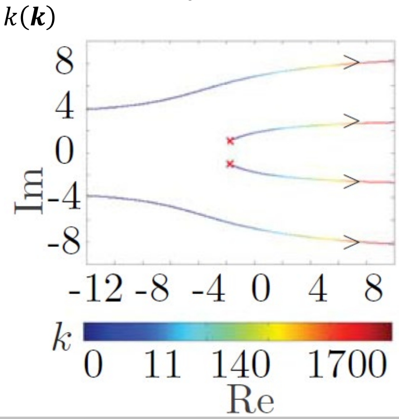
$$\varphi\big(k(\boldsymbol{k})\big) + \varphi\big(N_0(s)\big) - \varphi\big(D_0(s)\big) = \varphi(-1)$$
$$\varphi\ (N_0(s)) - \varphi(D_0(s)) = n180°$$

Note, the phase condition is independent of the controller gain $k(\boldsymbol{k})$.

The **magnitude condition** is used to parametrize the root locus curve with the varying parameter $k(\boldsymbol{k})$:

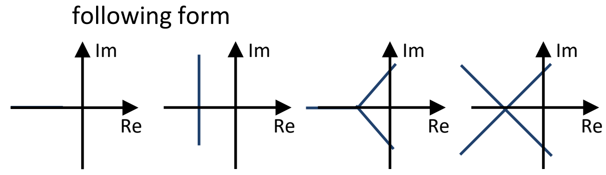
$$|k(\boldsymbol{k})| = 1/|\ \hat{G}_0(s)| = \frac{1}{\left|\frac{N_0(s)}{D_0(s)}\right|} = \frac{|D_0(s)|}{|N_0(s)|}$$

with $|k(\boldsymbol{k})| = k(\boldsymbol{k})$

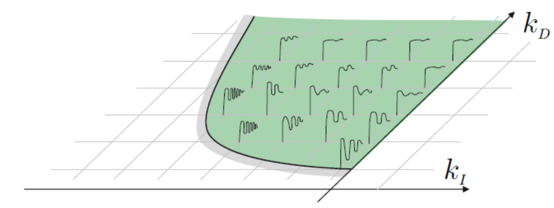


PLOTTING GUIDE (for more details, see e.g.[3,4])

1. The loci starts at the poles (x) of G_0 (with $k(\boldsymbol{k}) = 0$) and terminate (with $k(\boldsymbol{k}) \to \infty$) either at the zeros (o) of G_0 or at infinity.
2. For a G_0 with n poles and m zeros, $\mathrm{n}-\mathrm{m}$ loci terminate at infinity.
3. The number of loci equals the number of poles of G_0.
4. The root locus is symmetric about the real axis.
5. The root locus exist on the real axis only to the left of an odd number (for $k(\boldsymbol{k}) > 0$) of poles and zeros. In case of ($k(\boldsymbol{k}) < 0$) results a loci from an even number of poles and zeros.
6. The location of breakaway and break-in points are found by determining where the parameter $k(\boldsymbol{k})$ attains a local maximum or minimum on the real axis.
7. The loci that do not terminate at zero approach infinity along asymptotes. The asymptotes intersect the real axis at
$$s_A = \frac{\sum_{i=1}^{n} s_{Pi} - \sum_{i=1}^{m} s_{Zi}}{n-m}$$
They intersect the real axis under the angle
$$\alpha_i = \begin{cases} \frac{2i-1}{n-m} 180° \text{ für } k(\boldsymbol{k}) > 0 \\ \frac{2i}{n-m} 180° \text{ für } k(\boldsymbol{k}) < 0 \end{cases}$$
with $i = 1,2,\ldots,(n-m)$.
8. Loci can cross each other. The necessary condition for intersection points s_I is
$$\sum\nolimits_{i=1}^{n} \frac{1}{s_A - s_{Pi}} = \sum\nolimits_{i=1}^{m} \frac{1}{s_A - s_{Zi}}$$
If only two loci intersect, this is realized with an angle of 90°.
9. Angles of departure and angles of arrival are determined by choosing an arbitrary point close to the pole or zero and apply the angle criterion. For a departure point holds
$$\varphi_D = \sum\nolimits_{i=1}^{m} s_{Zi} - \sum\nolimits_{i=1}^{n} s_{Pi} - \begin{cases} \pi + q\ 2\pi \text{ for } k(\boldsymbol{k}) > 0 \\ 0 + q\ 2\pi\ \text{ for } k(\boldsymbol{k}) < 0 \end{cases}$$
For an arrival point holds
$$\varphi_A = \sum\nolimits_{i=1}^{n} s_{Pi} - \sum\nolimits_{i=1}^{m} s_{Zi} - \begin{cases} \pi + q\ 2\pi \text{ for } k(\boldsymbol{k}) > 0 \\ 0 + q\ 2\pi\ \text{ for } k(\boldsymbol{k}) < 0 \end{cases}$$
$q = 0, \pm 1, \ldots$
10. The intersection point of loci and imaginary axis can be evaluated by using the stability-boundary calculation.
11. Depending on the number of poles and zero, the asymptotes of the root locus can have the following form



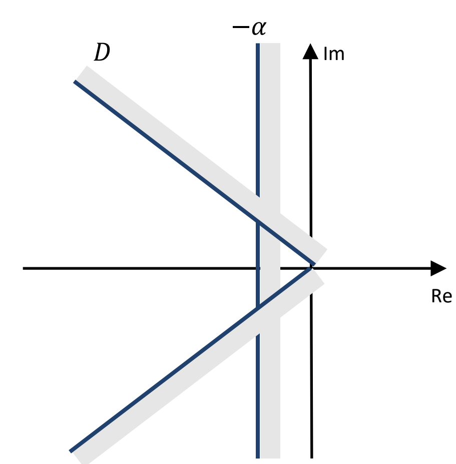
12. Finally the locus can be scaled with the magnitude criterion
$$|k(\boldsymbol{k})| = \frac{\prod_{i=1}^{n} |s-s_{Pi}|}{\prod_{i=1}^{m} |s-s_{Zi}|} \text{ and } |k(\boldsymbol{k})| = \begin{cases} 0 \text{ in } s_{Pi} \\ \infty \text{ in } s_{Zi} \end{cases}$$



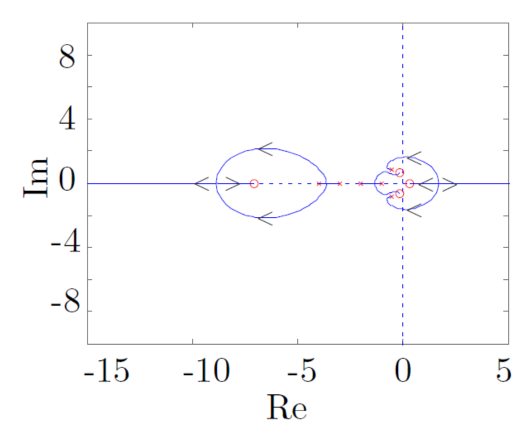
MAPPING PERFORMANCE CRITERIA

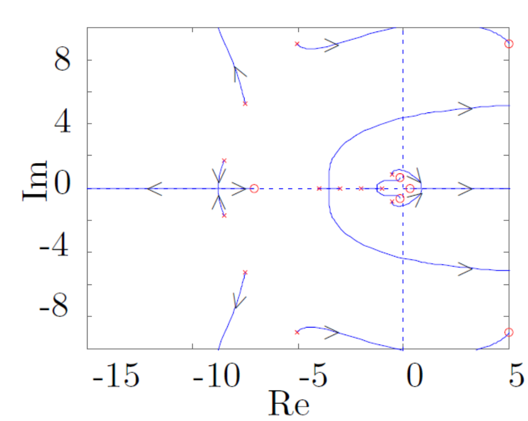
The **damping ratio** ζ can be mapped as two lines into the complex s-plane, passing the origin and making an angle of $\pm \cos^{-1} \zeta$.

In order to guarantee a **transient response** faster than $e^{-\alpha t}$, the closed loop poles should be located on the left hand side of the transient response line. These line is located parallel to the imaginary axis and cuts the real axis at the point α.



LOKI EXAMPLES





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[1] W. R. Evans. Graphical analysis of control systems. Transactions of the American Institute of Electrical Engineers, 67(1):547–551, 1948.
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